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Score-driven non-linear multivariate dynamic location models

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Abstract

In this paper, we introduce a new model by extending the dynamic conditional score (DCS) model of the multivariate t-distribution and name it as the quasi-vector autoregressive (QVAR) model. QVAR is a score-driven nonlinear multivariate dynamic location model, in which the conditional score vector of the log-likelihood (LL) updates the dependent variables. For QVAR, we present the details of the econometric formulation, the computation of the impulse response function, and the maximum likelihood (ML) estimation and related conditions of consistency and asymptotic normality. As an illustration, we use quarterly data for period 1987:Q1 to 2013:Q2 from the following variables: quarterly percentage change in crude oil real price, quarterly United States (US) inflation rate, and quarterly US real gross domestic product (GDP) growth. We find that the statistical performance of QVAR is superior to that of VAR and VARMA. Interestingly, stochastic annual cyclical effects with decreasing amplitude are found for QVAR, whereas those cyclical effects are not found for VAR or VARMA.

Keywords: Dynamic conditional score (DCS) models; multivariate dynamic location models; Quasi-VAR (QVAR) models; non-linear vector MA models, impulse response function (IRF), cyclical IRF, multivariate Student's t errors.

JEL codes: C32, C52

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1. Introduction

In this paper, we extend the dynamic conditional score (DCS) model of the multivariate t-distribution that was introduced in the work of Harvey (2013, Chapter 7.2.2). Motivated by Harvey (2013, Chapter 3.2.1), we name the new model as the quasi-vector autoregressive (QVAR) model. QVAR with lag-order p, denoted as QVAR(p), is a score-driven nonlinear multivariate dynamic location model, in which the conditional score vector of the log-likelihood (LL) updates the dependent variables. QVAR(p) is an extension of the DCS model for the multivariate t-distribution that is QVAR(1) under our notation. For QVAR, we present the details of the econometric formulation, the computation of the impulse response function, and the maximum likelihood (ML) estimation and related conditions of consistency and asymptotic normality.

As an illustration, we use quarterly macroeconomic time-series data for period 1987:Q1 to 2013:Q2 from the following $I(0)$ variables: (i) quarterly percentage change in crude oil real price; (ii) quarterly United States (US) inflation rate; (iii) quarterly US real gross domestic product (GDP) growth. For these data, all multivariate dynamic location models of this paper can be identified recursively. Thus, those models can be estimated by using short-run identifying restrictions for estimation (Kilian and Lütkepohl 2017, Chapters 8 and 9). However, it is important to note that the QVAR model suggested in this paper can also be applied for both $I(0)$ and possibly cointegrated $I(1)$ variables, for which either only long-run or both short-run and long-run identifying restrictions are used (Kilian and Lütkepohl 2017, Chapters 10 to 12).

We compare the statistical performance of QVAR and that of two benchmark multivariate dynamic location models: VAR and VARMA (vector autoregressive moving average). We estimate QVAR by using the ML method. We estimate VAR and VARMA by using the quasi-ML (QML) method. The likelihood-based model performance metrics suggest that the statistical performance of QVAR is superior to that of VAR and VARMA. The residual and conditional score diagnostic test results suggest that each residual and conditional score variable of QVAR(2) forms a multivariate i.i.d. time series. The conditions of consistency and asymptotic normality of ML are satisfied for QVAR.

Interestingly, stochastic annual cyclical effects with decreasing amplitude are found for QVAR, whereas those cyclical effects are not found for VAR or VARMA. For QVAR(2), a positive oil price shock generates positive persistent annual cyclical effects on inflation with decreasing amplitude, and persistent but oscillatory (negative and positive) annual cyclical effects on GDP growth with decreasing amplitude. For VAR(2) and VARMA(2,1), a positive oil price shock generates positive persistent and decreasing effects on inflation, and persistent and decreasing but negative effects on GDP growth.

The remainder of this paper is organized as follows. Section 2 presents the score-driven nonlinear multivariate dynamic location model. Section 3 presents the benchmark linear multivariate dynamic location model. Section 4 presents the identification of QVAR, VAR and VARMA. Section 5 presents the empirical results. Section 6 concludes.

2. Score-driven nonlinear multivariate dynamic location model: QVAR(p)

2.1. Model formulation

The reduced-form representation of QVAR(p) for y_t $(K \times 1)$ is

$$y_t = c + \mu_t + v_t \tag{1}$$

$$\mu_t = \Phi_1 \mu_{t-1} + \ldots + \Phi_p \mu_{t-p} + \Psi_1 u_{t-1} \tag{2}$$

where c $(K \times 1)$, $\Phi_1, \ldots, \Phi_p$ (each $K \times K$) and Ψ_1 $(K \times K)$ are time-constant parameters. For QVAR(p), the conditional expectation of y_t is $E(y_t|y_1, \ldots, y_{t-1}) = c + \mu_t$ $(K \times 1)$. For the first p observations, we initialize μ_t by using its unconditional mean $\mu_t = E(\mu_t) = 0_{K\times 1}$.

With respect to the updating terms, v_t $(K \times 1)$ is the reduced-form error term that updates y_t, and u_t $(K \times 1)$ is a vector of scaled score functions and its first lag updates μ_t. The reduced-form error term v_t is multivariate i.i.d. with $v_t \sim t_K(0, \Sigma, \nu)$, where $\Sigma = \Omega^{-1}(\Omega^{-1})'$ is positive definite and $\nu > 2$ denotes the degrees of freedom parameter ($\nu > 2$ ensures that the variance of v_t is finite). Under this assumption, the log of the conditional density of y_t is

$$\ln f(y_t|y_1,\ldots,y_{t-1}) = \ln\Gamma\left(\frac{\nu+K}{2}\right) - \ln\Gamma\left(\frac{\nu}{2}\right) - \frac{K}{2}\ln(\pi\nu) \tag{3}$$

$$-\frac{1}{2}\ln|\Sigma| - \frac{\nu+K}{2}\ln\left(1+\frac{v_t'\Sigma^{-1}v_t}{\nu}\right)$$

The partial derivative of the log of the conditional density with respect to μ_t is

$$\frac{\partial \ln f(y_t|y_1,\ldots,y_{t-1})}{\partial\mu_t} = \frac{\nu+K}{\nu}\Sigma^{-1}\times\left(1+\frac{v_t'\Sigma^{-1}v_t}{\nu}\right)^{-1}v_t = \frac{\nu+K}{\nu}\Sigma^{-1}\times u_t \tag{4}$$

where the last equality defines the score function u_t, which is the representation of u_t by using the reduced-form error term. Harvey (2013, Chapter 7) shows that u_t is multivariate i.i.d. with mean zero and covariance matrix

$$\mathrm{Var}(u_t) = E\left[\frac{\partial \ln f(y_t|y_1,\ldots,y_{t-1})}{\partial\mu_t}\times\frac{\partial \ln f(y_t|y_1,\ldots,y_{t-1})}{\partial\mu_t'}\right] = \frac{\nu+K}{\mu+K+2}\Sigma^{-1} \tag{5}$$

We use this property of u_t to undertake model diagnostics analysis after estimation.

With respect to the structural-form representation of QVAR(p), for v_t we have $E(v_t) = 0$ and $\mathrm{Var}(v_t) = \Sigma\times\nu/(\nu-2)$. We factorize $\mathrm{Var}(v_t)$ as

$$\mathrm{Var}(v_t) = \Sigma\times\frac{\nu}{\nu-2} = \left(\frac{\nu}{\nu-2}\right)^{1/2}\times\Omega^{-1}(\Omega^{-1})'\times\left(\frac{\nu}{\nu-2}\right)^{1/2} \tag{6}$$

and we introduce the multivariate i.i.d. structural-form error term ϵ_t as

$$v_t = \left(\frac{\nu}{\nu-2}\right)^{1/2}\Omega^{-1}\times\epsilon_t \tag{7}$$

where $E(\epsilon_t) = 0$, $\mathrm{Var}(\epsilon_t) = I_K$ and $\epsilon_t \sim t_K[0, I_K\times(\nu-2)/\nu, \nu]$ (see Kibria and Joarder 2006 about the linear transformation of multivariate t random variables). Therefore, the structural-form representation of QVAR is

$$\left(\frac{\nu}{\nu-2}\right)^{-1/2}\Omega y_t = \left(\frac{\nu}{\nu-2}\right)^{-1/2}\Omega c + \left(\frac{\nu}{\nu-2}\right)^{-1/2}\Omega \mu_t + \left(\frac{\nu}{\nu-2}\right)^{-1/2}\Omega v_t = \tag{8}$$

$$= \left(\frac{\nu}{\nu-2}\right)^{-1/2}\Omega c + \left(\frac{\nu}{\nu-2}\right)^{-1/2}\Omega \mu_t + \epsilon_t$$

Furthermore, we substitute Equation (7) into u_t which is defined in Equation (4) and obtain

$$u_t = [(\nu-2)\nu]^{1/2}\Omega^{-1} \times \frac{\epsilon_t}{\nu - 2 + \epsilon_t'\epsilon_t} \tag{9}$$

which is the representation of the score function u_t by using the structural-form error term.

In summary, the reduced-form QVAR(p) is given by

$$y_t = c + \Phi_1\mu_{t-1} + \ldots + \Phi_p\mu_{t-p} + \Psi_1 u_{t-1} + v_t \tag{10}$$

where the reduced-form error v_t has $E(v_t) = 0$ and $\mathrm{Var}(v_t) = [\nu/(\nu-2)]\Omega^{-1}(\Omega^{-1})'$. The structural-form QVAR(p) is given by

$$B_0 y_t = B_0 c + B_0\Phi_1\mu_{t-1} + \ldots + B_0\Phi_p\mu_{t-p} + B_0\Psi_1 u_{t-1} + \epsilon_t \tag{11}$$

where $B_0 = [\nu/(\nu-2)]^{-1/2}\Omega$, and the standard-form error has $E(\epsilon_t) = 0$ and $\mathrm{Var}(\epsilon_t) = I_K$. That is $v_t = B_0^{-1}\epsilon_t = [\nu/(\nu-2)]^{1/2}\Omega^{-1}\epsilon_t$.

2.2. Nonlinear vector MA representations and impulse response functions

The first-order representation of the reduced-form QVAR(p) of Equations (1) and (2) is

$$Y_t = C + M_t + V_t \tag{12}$$

$$M_t = \Phi M_{t-1} + \Psi U_{t-1} \tag{13}$$

where

$$
Y_t = \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{bmatrix}_{(Kp\times 1)} \qquad C = \begin{bmatrix} c \\ c \\ \vdots \\ c \end{bmatrix}_{(Kp\times 1)} \qquad M_t = \begin{bmatrix} \mu_t \\ \mu_{t-1} \\ \vdots \\ \mu_{t-p+1} \end{bmatrix}_{(Kp\times 1)} \qquad V_t = \begin{bmatrix} v_t \\ v_{t-1} \\ \vdots \\ v_{t-p+1} \end{bmatrix}_{(Kp\times 1)}
$$

$$
\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_{p-1} & \Phi_p \\ I_K & 0_{K\times K} & \cdots & \cdots & 0_{K\times K} \\ 0_{K\times K} & I_K & 0_{K\times K} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{K\times K} & \cdots & 0_{K\times K} & I_K & 0_{K\times K} \end{bmatrix}_{(Kp\times Kp)}
$$

$$
\Psi = \begin{bmatrix} \Psi_1 & 0_{K\times K} & \cdots & 0_{K\times K} \\ 0_{K\times K} & 0_{K\times K} & \cdots & 0_{K\times K} \\ \cdots & \cdots & \cdots & \cdots \\ 0_{K\times K} & \cdots & \cdots & 0_{K\times K} \end{bmatrix}_{(Kp\times Kp)} \qquad U_{t-1} = \begin{bmatrix} u_{t-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(Kp\times 1)}
$$

The reduced-form nonlinear vector MA(∞) representation of y_t is

$$
y_t = c + \left(\sum_{j=0}^{\infty} J\Phi^j J'\Psi_1 u_{t-1-j} \right) + v_t \tag{14}
$$

$$
y_t = c + \left[\sum_{j=0}^{\infty} J\Phi^j J'\Psi_1 \left(1 + \frac{v'_{t-1-j}\Sigma^{-1}v_{t-1-j}}{\nu} \right)^{-1} \right] + v_t \tag{15}
$$

The corresponding structural-form nonlinear vector MA(∞) representation of y_t is

$$
y_t = c + \left\{ \sum_{j=0}^{\infty} J\Phi^j J'\Psi_1 [(\nu-2)\nu]^{1/2} \Omega^{-1} \frac{\epsilon_{t-1-j}}{\nu - 2 + \epsilon'_{t-1-j}\epsilon_{t-1-j}} \right\} + \left(\frac{\nu}{\nu-2} \right)^{1/2} \Omega^{-1}\epsilon_t \tag{16}
$$

where $J = (I_K, 0_{K\times K}, \cdots, 0_{K\times K})$ $(K \times Kp)$. We use C_1 to denote the maximum modulus of all eigenvalues of Φ. $C_1 < 1$ implies that the series in Equations (14) to (16) are convergent.

The impulse response function $\text{IRF}_j = \partial y_{t+j} / \partial \epsilon_t$ for $j = 0, 1, \ldots, \infty$ is given by

$$\text{IRF}_0 = \left(\frac{\nu}{\nu - 2} \right)^{1/2} \Omega^{-1} \tag{17}$$

$$\text{IRF}_{jt} = J \Phi^j J' \Psi_1 [(\nu - 2)\nu]^{1/2} \Omega^{-1} D_{t-1-j} \quad \text{for} \quad j = 1, \ldots, \infty \tag{18}$$

where

$$D_t = \frac{\partial \frac{\epsilon_t}{\nu - 2 + \epsilon_t' \epsilon_t}}{\partial \epsilon_t} = \begin{bmatrix} d_{11,t} & \cdots & d_{1K,t} \\ \cdots & \cdots & \cdots \\ d_{K1,t} & \cdots & d_{KK,t} \end{bmatrix} = \tag{19}$$

$$= \begin{bmatrix} \frac{\nu - 2 + \epsilon_t' \epsilon_t - 2\epsilon_{1t}^2}{(\nu - 2 + \epsilon_t' \epsilon_t)^2} & \frac{-2\epsilon_{1t}\epsilon_{2t}}{(\nu - 2 + \epsilon_t' \epsilon_t)^2} & \cdots & \frac{-2\epsilon_{1t}\epsilon_{Kt}}{(\nu - 2 + \epsilon_t' \epsilon_t)^2} \\ \frac{-2\epsilon_{2t}\epsilon_{1t}}{(\nu - 2 + \epsilon_t' \epsilon_t)^2} & \frac{\nu - 2 + \epsilon_t' \epsilon_t - 2\epsilon_{2t}^2}{(\nu - 2 + \epsilon_t' \epsilon_t)^2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{-2\epsilon_{Kt}\epsilon_{1t}}{(\nu - 2 + \epsilon_t' \epsilon_t)^2} & \cdots & \cdots & \frac{\nu - 2 + \epsilon_t' \epsilon_t - 2\epsilon_{Kt}^2}{(\nu - 2 + \epsilon_t' \epsilon_t)^2} \end{bmatrix}$$

As IRF_{jt} for $j = 1, 2, \ldots, \infty$ depends on t, we evaluate its unconditional mean

$$\text{IRF}_j = E(\text{IRF}_{jt}) = J \Phi^j J' \Psi_1 [(\nu - 2)\nu]^{1/2} \Omega^{-1} E(D_{t-1-j}) \quad \text{for} \quad j = 1, 2, \ldots, \infty \tag{20}$$

If all elements of D_t form covariance stationary time series, then $E(D_{t-1-j})$ can be estimated by using the sample average (Hamilton 1994, Chapter 7.2). We test covariance stationarity of D_t by using the augmented Dickey–Fuller (1979) (ADF) unit root test with constant.

2.3. Maximum likelihood estimation

The parameters of QVAR are c, $\Phi_1, \ldots, \Phi_p$, Ψ_1, Ω^{-1} and ν. We estimate those parameters by using the ML method (Davidson and MacKinnon 2003). The ML estimator of parameters is

$$\hat{\Theta}_{\mathrm{ML}} = \arg\max_{\Theta} \mathrm{LL}(y_1, \ldots, y_T) = \arg\max_{\Theta} \sum_{t=1}^{T} \ln f(y_t | y_1, \ldots, y_{t-1}) \tag{21}$$

where Θ denotes the vector of time-constant parameters. We use the inverse information matrix to estimate the standard errors of parameters. We use the results from the work of Harvey (2013, Chapters 2.3, 2.4 and 3.3) to find the conditions under which the ML estimates of QVAR(p) are consistent and asymptotically Gaussian.

First, Condition 1 is $C_1 < 1$, which ensures that μ_t is covariance stationary. Second, we use Condition 2 from the work of Harvey (2013, p. 35, Condition 2). Condition 2 is that the score function u_t ($K \times 1$) and its first derivative $\partial u_t / \partial \mu_t$ ($K \times K$) have finite second moments and covariance that are time-invariant and do not depend on μ_t. To formalize this condition, we refer to the specific elements $u_{j,t}$ and $\partial u_{k,t} / \partial \mu_{l,t}$, where $j, k, l = 1, \ldots, K$. Condition 2 holds if $E[u_{j,t}^{2-i} (\partial u_{k,t} / \partial \mu_{l,t})^i] < \infty$, where $i = 0, 1, 2$ and $j, k, l = 1, \ldots, K$. We test Condition 2 by using the ADF test with constant for each $u_{j,t}^{2-i} (\partial u_{k,t} / \partial \mu_{l,t})^i$.

Third, in order to obtain Conditions 3 and 4 for QVAR(p), we use the arguments of the proof of Theorem 5 from the work of Harvey (2013, p. 49). We consider the representative element Ψ_{ij} from the matrix Ψ. From Equation (13), we have

$$\frac{\partial M_t}{\partial \Psi_{ij}} = \Phi \frac{\partial M_{t-1}}{\partial \Psi_{ij}} + \Psi \frac{\partial U_{t-1}}{\partial \Psi_{ij}} + W_{ij} U_{t-1} \tag{22}$$

for all $t = 1, \ldots, T$, where the element (i, j) of the matrix W_{ij} ($Kp \times Kp$) is one and the rest of the elements of W_{ij} are zero. We use the chain rule to express

$$\frac{\partial U_{t-1}}{\partial \Psi_{ij}} = \frac{\partial U_{t-1}}{\partial M'_{t-1}} \frac{\partial M_{t-1}}{\partial \Psi_{ij}} \tag{23}$$

and we substitute this equation into Equation (22) to get the first-order AR representation

$$\frac{\partial M_t}{\partial \Psi_{ij}} = \left(\Phi + \Psi \frac{\partial U_{t-1}}{\partial M'_{t-1}}\right) \frac{\partial M_{t-1}}{\partial \Psi_{ij}} + W_{ij} U_{t-1} = X_t \frac{\partial M_{t-1}}{\partial \Psi_{ij}} + W_{ij} U_{t-1} \tag{24}$$

where X_t $(Kp \times Kp)$ is defined by the last equality. Condition 3 is that all eigenvalues of $E(X_t)$ are within the unit circle. We denote the maximum modulus of all eigenvalues of $E(X_t)$ by using C_3. If each element of X_t is covariance stationary, then $E(X_t)$ can be estimated by using the sample average. We test covariance stationarity of X_t by using the ADF test with constant. Based on the arguments of Harvey (2013, p. 49), Condition 3 is a necessary condition of consistency and asymptotic normality of ML.

Furthermore, the information matrix of QVAR(p) depends on the following term, expressed using Equation (24) for the specific elements (i, j) and (k, l):

$$\frac{\partial M_t}{\partial \Psi_{ij}} \frac{\partial M'_t}{\partial \Psi_{kl}} = X_t \frac{\partial M_{t-1}}{\partial \Psi_{ij}} \frac{\partial M'_{t-1}}{\partial \Psi_{kl}} X'_t + X_t \frac{\partial M_{t-1}}{\partial \Psi_{ij}} W'_{ij} U_{t-1} + U'_{t-1} W_{kl} \frac{\partial M'_{t-1}}{\partial \Psi_{kl}} X'_t + W_{ij} U_{t-1} U'_{t-1} W'_{kl} \tag{25}$$

We can write this equation as the first-order dynamic representation

$$\text{vec}\left(\frac{\partial M_t}{\partial \Psi_{ij}} \frac{\partial M'_t}{\partial \Psi_{kl}}\right) = (X_t \otimes X_t) \text{vec}\left(\frac{\partial M_{t-1}}{\partial \Psi_{ij}} \frac{\partial M'_{t-1}}{\partial \Psi_{kl}}\right) + \tag{26}$$

$$+ \text{vec}\left(X_t \frac{\partial M_{t-1}}{\partial \Psi_{ij}} W'_{ij} U_{t-1}\right) + \text{vec}\left(U'_{t-1} W_{kl} \frac{\partial M'_{t-1}}{\partial \Psi_{kl}} X'_t\right) + \text{vec}\left(W_{ij} U_{t-1} U'_{t-1} W'_{kl}\right)$$

where $\otimes$ is the Kronecker product and vec(x) indicates that the columns of the matrix are being stacked one upon the other. Condition 4 is that all eigenvalues of $E(X_t \otimes X_t)$ are within the unit circle. We denote the maximum modulus of all eigenvalues of $E(X_t \otimes X_t)$ by using C_4. If each element of $X_t \otimes X_t$ is covariance stationary, then $E(X_t \otimes X_t)$ can be estimated by using the sample average. We test covariance stationarity of $X_t \otimes X_t$ by using the ADF test with constant. Based on the arguments of Harvey (2013, p. 49), Condition 4 is a sufficient condition of consistency and asymptotic normality of ML.

Is is noteworthy that for the computation of $X_t = \Phi + \Psi(\partial U_{t-1}/\partial M'_{t-1})$, we need the formula for $\partial u_t/\partial \mu'_t$ $(K \times K)$. As aforementioned, the score function is given by

$$u_t = \left(1 + \frac{v'_t \Sigma^{-1} v_t}{\nu}\right)^{-1} v_t = \frac{\nu(y_t - c - \mu_t)}{\nu + (y_t - c - \mu_t)'\Sigma^{-1}(y_t - c - \mu_t)} \tag{27}$$

and the formula of $\partial u_t/\partial \mu'_t$ can be obtained by using standard matrix calculus.

3. Benchmark linear multivariate dynamic location model: VARMA(p,1)

3.1. Model formulation

Motivated by the fact that the use of u_{t-1} in Equation (2) for QVAR(p) is similar to that of the MA term for VARMA$(p, 1)$, we use VARMA$(p, 1)$ as the benchmark linear multivariate dynamic location model. The reduced-form representation of VARMA$(p, 1)$ for y_t $(K \times 1)$ is

$$y_t = \mu_t + v_t = \mu_t + \Omega^{-1}\epsilon_t \tag{28}$$

where μ_t is the conditional mean of $y_t|(y_1, \ldots, y_{t-1})$ that is specified as

$$E(y_t|y_1, \ldots, y_{t-1}) = \mu_t = c + \Phi_1 y_{t-1} + \cdots + \Phi_p y_{t-p} + \Psi_1 v_{t-1} \tag{29}$$

where c $(K \times 1)$, $\Phi_1, \ldots, \Phi_p$ (each $K \times K$) and Ψ_1 $(K \times K)$ are time-constant parameters. Under the restriction $\Psi_1 = 0_{K\times K}$, we obtain the VAR(p) model. For the first p observations, we initialize μ_t by using the unconditional mean $\mu_t = E(y_t) = J(I_{Kp} - \Phi)^{-1}C$, where

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_{p-1} & \Phi_p \\ I_K & 0_{K\times K} & \cdots & \cdots & 0_{K\times K} \\ 0_{K\times K} & I_K & 0_{K\times K} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{K\times K} & \cdots & 0_{K\times K} & I_K & 0_{K\times K} \end{bmatrix}_{(Kp\times Kp)} \qquad C = \begin{bmatrix} c \\ 0_{K\times 1} \\ \cdots \\ 0_{K\times 1} \end{bmatrix}_{(Kp\times 1)}$$

and $J = [I_K, 0_{K\times K}, \cdots, 0_{K\times K}]$ $(K \times Kp)$. We denote the maximum modulus of all eigenvalues of Φ by using C_1. Furthermore, v_t is the multivariate i.i.d. reduced-form error term with mean $E(v_t) = 0_{K\times 1}$ and positive definite covariance matrix $\mathrm{Var}(v_t) = \Sigma_{K\times K} = \Omega^{-1}(\Omega^{-1})'$. The multivariate i.i.d. structural-form error term $\epsilon_t = \Omega v_t$ has mean $E(\epsilon_t) = 0_{K\times 1}$ and covariance matrix $\mathrm{Var}(\epsilon_t) = I_K$. The structural-form representation of VARMA is

$$\Omega y_t = \Omega \mu_t + \Omega v_t = \Omega \mu_t + \epsilon_t \tag{30}$$

and the structural-form MA(∞) representation of y_t is

$$y_t = \sum_{j=0}^{\infty} \Phi_1^j c + \sum_{j=0}^{\infty} \Pi_j \Omega^{-1} \epsilon_{t-j} \tag{31}$$

where

$$\begin{aligned}
\Pi_0 &= I_K \\
\Pi_1 &= \Psi_1 + \Phi_1 \Pi_0 \\
\Pi_2 &= \Phi_1 \Pi_1 + \Phi_2 \Pi_0 \\
&\vdots \\
\Pi_p &= \Phi_1 \Pi_{p-1} + \ldots + \Phi_p \Pi_0 \\
\Pi_j &= \Phi_1 \Pi_{j-1} + \ldots + \Phi_p \Pi_{j-p} \quad \text{for } j > p
\end{aligned}$$

Both series in Equation (31) are convergent if $C_1 < 1$ (Lütkepohl 2005, Chapter 11). The impulse response function is given by $\mathrm{IRF}_j = \partial y_{t+j}/\partial \epsilon_t = \Pi_j \Omega^{-1}$ for $j = 0, 1, \ldots, \infty$.

3.2. Quasi-maximum likelihood estimation

The parameters of VARMA are c, $\Phi_1, \ldots, \Phi_p$, Ψ_1 and Ω^{-1}. We estimate those parameters by using the QML method (Gouriéroux et al. 1984a, 1984b), for which we use the pseudo probability distribution $v_t \sim N_K(0, \Sigma)$ for the reduced-form error term. Hence, the log of the pseudo conditional density of y_t is

$$\ln f(y_t | y_1, \ldots, y_{t-1}) = -\frac{1}{2} \ln |2\pi\Sigma| - \frac{1}{2} v_t' \Sigma^{-1} v_t \tag{32}$$

The QML estimates of parameters is given by:

$$\hat{\Theta}_{\mathrm{QML}} = \arg\max_{\Theta} \mathrm{LL}(y_1, \ldots, y_T) = \arg\max_{\Theta} \sum_{t=1}^{T} \ln f(y_t | y_1, \ldots, y_{t-1}) \tag{33}$$

The condition $C_1 < 1$ ensures that y_t is covariance stationary (Lütkepohl 2005, Chapter 11). Moreover, we denote the maximum modulus of all eigenvalues of Ψ_1 by using C_2. The condition $C_2 < 1$ ensures that all eigenvalues of Ψ_1 are inside the unit circle, and that the VARMA$(p, 1)$ process is invertible (Lütkepohl 2005, Chapter 11). For VARMA$(p, 1)$, the conditions $C_1 < 1$ and $C_2 < 1$ ensure that QML is consistent and asymptotically normal (Lütkepohl 2005, Chapter 12). For VAR(p), the condition $C_1 < 1$ ensures that QML is consistent and asymptotically normal.

3.3. Comparison of VARMA$(p, 1)$ and QVAR(p)

We finish this section by comparing the dynamic specifications of VARMA$(p, 1)$ and QVAR(p). Substituting Equation (1) into Equation (2), we obtain the following representation of QVAR(p):

$$y_t = c + \Phi_1 y_{t-1} + \ldots + \Phi_p y_{t-p} + v_t \underbrace{-\Phi_1 v_{t-1}}_{*} \tag{34}$$

$$\underbrace{-\Phi_1 c - \ldots - \Phi_p c - \Phi_2 v_{t-2} - \ldots - \Phi_p v_{t-p} + \Psi_1 u_{t-1}}_{**}$$

From this representation, we obtain VARMA$(p, 1)$ of Equations (28) and (29), by replacing the parameter of the term indicated by $*$ with parameter Ψ_1 (i.e., $-\Phi_1 = \Psi_1$), and by excluding the terms indicated by $**$. Thus, QVAR(p) is an alternative nonlinear version of VARMA$(p, 1)$, since: (i) It includes an additional score function term $\Psi_1 u_{t-1}$ that is a nonlinear transformation of v_{t-1}; (ii) It includes $p - 1$ additional MA terms with parameters $-\Phi_2, \ldots, -\Phi_p$.

4. Identification of QVAR, VAR and VARMA

4.1. Identification of structural forms

The QVAR, VAR and VARMA models used in this paper are recursively identified structural models (Kilian and Lütkepohl 2017, Chapter 9). This identification method is supported by the argument that oil price shocks may act as domestic supply shocks for the US economy (Kilian and Lütkepohl 2017, p. 239).

In the following, we present the identification of the most general QVAR model of this paper: QVAR(2) (for this model, estimation results are reported in Section 5). It is noteworthy that the identification of the structural-form representation is identical for all other models of this paper. Let $K = 3$ and $P = 2$ in Equations (1) and (2), then the reduced-form QVAR(2) is

$$\begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} \Phi_{1,11} & \Phi_{1,12} & \Phi_{1,13} \\ \Phi_{1,21} & \Phi_{1,22} & \Phi_{1,23} \\ \Phi_{1,31} & \Phi_{1,32} & \Phi_{1,33} \end{bmatrix} \begin{bmatrix} \mu_{1,t-1} \\ \mu_{2,t-1} \\ \mu_{3,t-1} \end{bmatrix} + \tag{35}$$

$$+ \begin{bmatrix} \Phi_{2,11} & \Phi_{2,12} & \Phi_{2,13} \\ \Phi_{2,21} & \Phi_{2,22} & \Phi_{2,23} \\ \Phi_{2,31} & \Phi_{2,32} & \Phi_{2,33} \end{bmatrix} \begin{bmatrix} \mu_{1,t-2} \\ \mu_{2,t-2} \\ \mu_{3,t-2} \end{bmatrix} + \begin{bmatrix} \Psi_{1,11} & \Psi_{1,12} & \Psi_{1,13} \\ \Psi_{1,21} & \Psi_{1,22} & \Psi_{1,23} \\ \Psi_{1,31} & \Psi_{1,32} & \Psi_{1,33} \end{bmatrix} \begin{bmatrix} u_{1,t-1} \\ u_{2,t-1} \\ u_{3,t-1} \end{bmatrix} + \begin{bmatrix} v_{1t} \\ v_{2t} \\ v_{3t} \end{bmatrix}$$

Let $\mathrm{Var}(v_t) = [\nu/(\nu-2)] \times \Sigma = [\nu/(\nu-2)] \times \Omega^{-1}(\Omega^{-1})'$ where

$$\Omega^{-1} = \begin{bmatrix} \Omega^{-1}_{11} & 0 & 0 \\ \Omega^{-1}_{21} & \Omega^{-1}_{22} & 0 \\ \Omega^{-1}_{31} & \Omega^{-1}_{32} & \Omega^{-1}_{33} \end{bmatrix} \tag{36}$$

is a lower-triangular matrix. Then,

$$v_t = \left(\frac{\nu}{\nu-2}\right)^{1/2} \Omega^{-1}\epsilon_t = \left(\frac{\nu}{\nu-2}\right)^{1/2} \begin{bmatrix} \Omega^{-1}_{11}\epsilon_{1t} \\ \Omega^{-1}_{21}\epsilon_{1t} + \Omega^{-1}_{22}\epsilon_{2t} \\ \Omega^{-1}_{31}\epsilon_{1t} + \Omega^{-1}_{32}\epsilon_{2t} + \Omega^{-1}_{33}\epsilon_{3t} \end{bmatrix} \tag{37}$$

As a consequence, for QVAR, VAR and VARMA, the decomposition $\Sigma = \Omega^{-1}(\Omega^{-1})'$ is a Cholesky decomposition. As Σ is positive definite for all cases, the Cholesky decomposition is unique if the diagonal of Ω^{-1} includes positive elements (i.e., $\Omega^{-1}_{11} > 0$, $\Omega^{-1}_{22} > 0$ and $\Omega^{-1}_{33} > 0$).

4.2. Identification of model parameters

For QVAR(p) of Section 2, the ML procedure did not converge to an optimum for the dataset used in this paper. Therefore, we use the $\Psi_1 = \Psi_{1,11} \times I_K$ restriction, where $\Psi_{1,11} \in \mathbb{R}$. This implies that Ψ_1 is a diagonal matrix with $\Psi_{1,11} = \Psi_{1,22} = \Psi_{1,33}$. Under this restriction, we identify all elements of parameters c, Φ, Ω^{-1} and ν. In this paper, we report results for the corresponding QVAR(1) and QVAR(2) specifications.

For VARMA(p,1) of Section 3, the QML procedure did not converge to an optimum for the dataset used in this paper. Therefore, we use the $\Psi_1 = \Psi_{1,11} \times I_K$ restriction, where $\Psi_{1,11} \in \mathbb{R}$. This implies that Ψ_1 is a diagonal matrix with $\Psi_{1,11} = \Psi_{1,22} = \Psi_{1,33}$. Under this restriction, we identify all elements of parameters c, Φ and Ω^{-1}. For VAR(p) of Section 3, the QML procedure converged to an optimum. In this paper, we report results for the corresponding VAR(1), VAR(2), VARMA(1,1) and VARMA(2,1) specifications.

5. Empirical results

5.1. Data

We use macroeconomic data from the book of Kilian and Lütkepohl (2017) (source: http://www-personal.umich.edu/~lkilian/figure9_1_chol.zip; accessed August 19, 2017). This dataset includes the following variables: (i) monthly West Texas Intermediate (WTI) price of crude oil for period December 1972 to June 2013; (ii) quarterly US GDP deflator for period 1959:Q1 to 2013:Q2; (iii) quarterly US real GDP level for period 1959:Q1 to 2013:Q2. The use of these variables is motivated by several works from the body of literature, which study the question of how oil price shocks affect US real GDP and inflation (e.g., Blanchard 2002; Barsky and Kilian 2004; Kilian 2008; Kilian and Lütkepohl 2017).

Similar to the work of Kilian and Lütkepohl (2017), we define: (i) variable y_{1t} as the quarterly first difference of log real price of crude oil (hereinafter, crude oil); (ii) variable y_{2t} as the quarterly first difference of log US GDP deflator (hereinafter, inflation); (iii) variable y_{3t} as the quarterly first difference of log US real GDP level (hereinafter, GDP growth). We define $y_t = (y_{1t}, y_{2t}, y_{3t})'$, hence, $K = 3$ for all models in this paper. Furthermore, we use data for period

1987:Q1 to 2013:Q2 (Kilian and Lütkepohl 2017, Chapter 9.2.1) (see Fig. 1). In this paper, all variables are measured in percentage points. Some descriptive statistics and tests of the dataset are presented in Table 1. The ADF test for three alternative specifications suggests that all dependent variables are $I(0)$ (Table 1). The partial autocorrelation function (PACF) estimates suggest significant serial correlation for several lags of the dependent variables (Table 1).

[APPROXIMATE LOCATION OF TABLE 1 AND FIGURE 1]

5.2. ML estimates and model diagnostics

We present the parameter estimates and model diagnostics for all models in Tables 2 and 3, respectively. In the following, we summarize the most important results.

First, we find that some elements of Φ and Ω^{-1} are significantly different from zero for all models (Table 2). This suggests significant dynamic and contemporaneous interaction effects, respectively, among crude oil, inflation and GDP growth. We study those interaction effects in Sections 5.3 and 5.4, by using the impulse response function.

Second, for both VAR specifications, the Ljung–Box (1978) (LB) test suggests that ϵ_{2t} and v_{2t} each form a non-independent time series (Table 3). For all VARMA and QVAR specifications, the LB test suggests that each reduced-form and structural-form error term forms an independent time series (Table 3). For QVAR(1), the LB test suggests that the score function u_{1t} is a non-independent time series (Table 3). For QVAR(2), we find that each score function forms an independent time series (Table 3). Thus, the residual- and score-related diagnostics support the VARMA(1,1), VARMA(2,1) and QVAR(2) specifications.

Third, for all models, we find that conditions of consistency and asymptotic normality of ML and QML are supported by the C_1, C_2, C_3 and C_4 metrics (Table 3). In the following, we review the related results for standard multivariate dynamic location specifications (VAR and VARMA) and score-driven multivariate dynamic location specifications (QVAR).

With respect to standard multivariate dynamic location specifications, both VAR(1) and VAR(2) are covariance stationary with $C_1 = 0.6017$ and $C_1 = 0.7779$, respectively (Table 3). Both VARMA(1,1) and VARMA(2,1) are covariance stationary with $C_1 = 0.9029$ and $C_1 =$

0.8874, respectively (Table 3). Moreover, both VARMA(1,1) and VARMA(2,1) are invertible with $C_2 = 0.5969$ and $C_2 = 0.5495$, respectively (Table 3).

With respect to score-driven multivariate dynamic location specifications, Condition 1 of QVAR is supported, since both QVAR(1) and QVAR(2) are covariance stationary with $C_1 = 0.8660$ and $C_1 = 0.8932$, respectively (Table 3).

For Condition 2 of QVAR, we perform the ADF test for the time series $u_{j,t}^{2-i}(\partial u_{k,t}/\partial \mu_{l,t})^i$ for $t = 1, \ldots, T$, where $i = 0, 1, 2$ and $j, k, l = 1, 2, 3$; that are in total $3K^3 = 81$ times series for both QVAR(1) and QVAR(2). For all time series of Condition 2, we find that the unit root null hypothesis of the ADF test is always rejected. We do not report the ADF results corresponding to Condition 2 in this paper, due to the large number of ADF test results. In Table 3, we summarize those ADF results with C_2 ADF.

Condition 3 of QVAR is supported, since $C_3 = 0.8230$ and $C_3 = 0.9010$ for QVAR(1) and QVAR(2), respectively (Table 3). Condition 4 of QVAR is supported, since $C_4 = 0.6791$ and $C_4 = 0.8137$ for QVAR(1) and QVAR(2), respectively (Table 3). We perform the ADF test for each time series defined by the elements of X_t (Condition 3) and $X_t \otimes X_t$ (Condition 4) for $t = 1, \ldots, T$. For Condition 3 we study $(Kp)^2$ time series; that are 9 time series for QVAR(1) and 36 time series for QVAR(2). For Condition 4 we study $(Kp)^4$ time series; that are 81 time series for QVAR(1) and 1,296 times series for QVAR(2). For all time series of Conditions 3 and 4, we find that the unit root null hypothesis of the ADF test is always rejected. We do not report the ADF results corresponding to Conditions 3 and 4 in this paper, due to the large number of ADF test results. In Table 3, we summarize those ADF results with C_3 ADF and C_4 ADF.

Fourth, we compare the statistical performance of QVAR, VAR and VARMA, by estimating the following likelihood-based performance metrics: (i) mean LL; (ii) mean Akaike information criterion (AIC); (iii) mean Bayesian information criterion (BIC); (iv) mean Hannan–Quinn criterion (HQC) (Davidson and MacKinnon 2003). All model performance metrics suggest that the statistical performance of QVAR is superior to that of VAR and VARMA (Table 3).

[APPROXIMATE LOCATION OF TABLES 2 AND 3]

5.3. Impulse response functions

From the empirical results reported in Table 2 we see that we can never reject the null hypothesis that $\Phi_{1,12} = \Phi_{1,13} = \Phi_{2,12} = \Phi_{2,13} = 0$. Since, by the Cholesky decomposition we impose that the oil price changes are predetermined with $\Omega_{12}^{-1} = \Omega_{13}^{-1} = 0$ (Kilian 2008) therefore we conclude that oil price changes are strictly exogenous (Kilian 2008) for the parameters of interest in the inflation and the GDP growth rate equations. Furthermore, since $\Omega_{21}^{-1} = 0$ therefore the inflation rate is also predetermined (Kilian 2008).

We present the impulse response functions of the QVAR(2), VAR(2) and VARMA(2,1) models in Figs. 2, 3 and 4, respectively. We present each impulse response function from 0 to 20 lags. In each figure, we present the mean of the impulse response function plus its 5% and 95% quantile levels estimated by using 10,000 simulations from the asymptotic distribution of the parameters. For QVAR, we estimate $E(D_t)$ of Equation (20), by using the sample average of D_t. We justify the use of this estimator by performing the ADF test with constant for each element of D_t (Hamilton 1994, Chapter 7.2). The ADF test results presented in Table 3 suggest that all elements of D_t are covariance stationary.

The most important finding from Figs. 2 to 4 is that stochastic annual cyclical effects are found for the QVAR(2) model, which are related to crude oil and GDP growth. Nevertheless, those cyclical effects are not identified for VAR(2) and VARMA(2,1). For QVAR(2), a positive oil price shock generates positive persistent annual cyclical effects with decreasing amplitude on inflation (Fig. 2(d)), and persistent but oscillatory (negative and positive) annual cyclical effects with decreasing amplitude on GDP growth (Fig. 2(g)). For VAR(2) and VARMA(2,1), the results are similar: a positive oil price shock generates positive persistent and decreasing effects on inflation (Fig. 3(d) and Fig. 4(d), respectively), and persistent and decreasing but negative effects on GDP growth (Fig. 3(g) and Fig. 4(g), respectively).

[APPROXIMATE LOCATION OF FIGURES 2 TO 4]

5.4. Robustness analysis

Perhaps, the most surprising result of the present paper is that for QVAR(2) stochastic annual cyclical effects are identified for the impulse response function. Based on the estimates presented in Table 2, one may argue that these cyclical effects may be spurious, because they may be due to the non-significant parameters within Φ_1, Φ_2 and Ω^{-1}, or they may be due to the order of variables in QVAR. In this section we investigate this concern.

First, we assume that the non-significant coefficients of Φ_1, Φ_2 and Ω^{-1} from Table 2 are equal to zero. We present the corresponding ML estimates and model diagnostics for the restricted QVAR(2) and VARMA(2,1) models in Table 4. We present the impulse response functions for the restricted QVAR(2) and VARMA(2,1) models in Figs. 5 and 6, respectively. We find that annual cyclical effects are identified for the impulse response function of the restricted QVAR(2) specification, hence, the results for QVAR(2) of Table 2 are robust.

Second, we estimate QVAR(2) (Table 2) and the restricted QVAR(2) (Table 4) for the alternative variable ordering of (i) crude oil, (ii) GDP growth and (iii) inflation. We present the ML estimation results and model diagnostics in Table 5. Furthermore, we present the corresponding impulse response functions in Figs. 7 and 8, respectively. For both cases we find that the same annual cyclical effects are identified for the impulse response function of QVAR(2), hence, the estimation results for QVAR(2) of Table 2 are robust. The IRF of shocks in inflation and GDP growth are very similar independent of the ordering of the variables (see Figs. 2, 5, 7 and 8). For example, from Figs. 5 and 8, we see that for QVAR(2) a shock in inflation (Fig. 5(h) and Fig. 8(f)) generates oscillatory (positive and negative) annual cyclical effects on GDP growth. Furthermore, a shock in GDP growth has positive annual cyclical effects on inflation with decreasing magnitude (see Fig. 5(f) and Fig. 8(h)). In the latter figures, we impose that the rate of change in oil price is strictly exogenous for inflation and GDP growth in the US.

[APPROXIMATE LOCATION OF TABLES 4-5 AND FIGURES 5-8]

6. Conclusions

In this paper, we have introduced the new QVAR(p) model. For this model, we have presented the details of the econometric formulation, the computation of the impulse response function, and the ML estimation and related conditions of consistency and asymptotic normality. We have used macroeconomic data from the book of Kilian and Lütkepohl (2017) for period 1987:Q1 to 2013:Q2. The variables considered have been quarterly percentage change in crude oil real price, quarterly US inflation rate, and quarterly US real GDP growth. We have found that the statistical performance of QVAR is superior to that of VAR and VARMA. Stochastic annual cyclical effects related to crude oil and GDP growth have been identified for QVAR(2), whereas the same cyclical effects have not been identified for VAR(2) and VARMA(2,1).

We have found that for QVAR(2) a positive oil price shock generates positive persistent annual cyclical effects with decreasing amplitude on inflation, and persistent but oscillatory (negative and positive) annual cyclical effects with decreasing amplitude on GDP growth. For VAR(2) and VARMA(2,1), the results are similar: a positive oil price shock generates positive persistent and decreasing effects on inflation, and persistent and decreasing but negative effects on GDP growth. The IRF of shocks in inflation and GDP growth are very similar independent of the ordering of the variables. For example, we have seen that for QVAR(2) a shock in inflation generates oscillatory (positive and negative) cyclical effects on GDP growth. Furthermore, a shock in GDP growth has positive annual cyclical effects on inflation with decreasing magnitude, by imposing that the rate of change in oil price is strictly exogenous for inflation and GDP growth in the US.

All multivariate dynamic location models of this paper have been estimated for $I(0)$ variables, by using short-run identifying restrictions only. Nevertheless, in future work, the QVAR model suggested in this paper can be also applied for $I(0)$ and possibly cointegrated $I(1)$ variables, for which either only long-run or both short-run and long-run identifying restrictions are used for estimation.

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Table 1. Descriptive statistics

	Crude oil y_{1t}	Inflation y_{2t}	GDP growth y_{3t}
Start date	1987:Q1	1987:Q1	1987:Q1
End date	2013:Q2	2013:Q2	2013:Q2
Sample size T	106	106	106
Minimum	−93.1315	−0.1560	−2.1746
Maximum	68.2694	1.2005	1.8712
Mean	1.1289	0.5548	0.6430
Standard deviation	17.6076	0.2423	0.6212
Skewness	−1.0045	0.2055	−1.3129
Excess kurtosis	8.3888	0.4469	4.0935
ADF with constant	−9.4521***	−2.8647**	−6.3930***
ADF with constant and linear trend	−9.4633***	−3.4689**	−6.7048***
ADF with constant and quadratic trend	−9.4560***	−3.5659*	−6.7431***
PACF(1)	−0.0238	0.6536***	0.4319***
PACF(2)	−0.263***	0.2862***	0.24**
PACF(3)	0.0209	0.1336	−0.0769
PACF(4)	−0.0407	0.0507	0.0806
PACF(5)	−0.2431**	−0.0418	−0.0463
PACF(6)	−0.0473	0.0859	0.0067
PACF(7)	0.0178	−0.1441	0.0306
PACF(8)	−0.1096	−0.0356	−0.0325
PACF(9)	−0.1454	−0.0469	0.1543
PACF(10)	−0.0401	−0.0035	−0.0603
PACF(11)	−0.076	0.0794	−0.2349**
PACF(12)	0.0923	−0.0351	−0.0327
PACF(13)	−0.169*	−0.1464	−0.0068
PACF(14)	0.1392	0.0772	0.1337
PACF(15)	−0.0532	−0.0354	0.0196
PACF(16)	0.0546	0.1065	0.0285
PACF(17)	−0.0422	−0.0276	−0.0064
PACF(18)	−0.0357	−0.1058	−0.1212
PACF(19)	0.0062	−0.1567	0.1495
PACF(20)	0.0228	0.1108	0.0017

Notes: Gross domestic product (GDP); augmented Dickey–Fuller test statistic (ADF); partial autocorrelation function (PACF).
*, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively.
Source of data: http://www-personal.umich.edu/∼lkilian/figure9_1_chol.zip. Accessed August 19, 2017.

Table 2. Parameter estimates; VAR(1), VAR(2), VARMA(1,1), VARMA(2,1), QVAR(1) and QVAR(2) models

	VAR(1)	VAR(2)	VARMA(1,1)	VARMA(2,1)	QVAR(1)	QVAR(2)
c_1	2.2474(7.1518)	−0.6156(6.3289)	0.4544(3.2069)	−0.3198(3.6659)	2.7357(1.7330)	3.6927(2.9517)
c_2	0.1359**(0.0640)	0.0719(0.0626)	0.0037(0.0241)	0.0086(0.0321)	0.5628***(0.0688)	0.5982***(0.0836)
c_3	0.4377**(0.2137)	0.4321**(0.2091)	0.1913**(0.0879)	0.2031*(0.1137)	0.6944***(0.0831)	0.6997***(0.0635)
$\Phi_{1,11}$	−0.0483(0.1323)	−0.0551(0.1381)	0.4148***(0.1526)	0.4535**(0.1885)	0.4059(0.2914)	0.0434(0.4088)
$\Phi_{1,12}$	−5.0643(11.3223)	−11.6530(14.6404)	−1.5651(5.0321)	−2.4832(16.8823)	−17.4749(11.5521)	−14.0767(17.4553)
$\Phi_{1,13}$	2.7481(3.5702)	2.8974(4.1635)	1.7358(1.8527)	2.8715(4.9101)	−10.9163(6.8459)	−14.9229(10.5581)
$\Phi_{1,21}$	−0.0001(0.0018)	0.0003(0.0018)	0.0017(0.0012)	0.0006(0.0018)	0.0083*(0.0045)	0.0093*(0.0057)
$\Phi_{1,22}$	0.6615***(0.0923)	0.4208***(0.1187)	0.9241***(0.0387)	0.8223***(0.2101)	1.1893***(0.1546)	1.6014***(0.1649)
$\Phi_{1,23}$	0.0776**(0.0367)	0.0698(0.0447)	0.0535***(0.0185)	0.0776*(0.0453)	0.2019**(0.0866)	0.2874***(0.1111)
$\Phi_{1,31}$	−0.0012(0.0031)	−0.0012(0.0028)	−0.0030(0.0028)	−0.0018(0.0029)	−0.0038(0.0092)	−0.0058(0.0156)
$\Phi_{1,32}$	−0.1380(0.3169)	−0.2377(0.4349)	−0.1899(0.1417)	−0.1016(0.4782)	−0.3238(0.3283)	0.6641(0.8137)
$\Phi_{1,33}$	0.4428***(0.1133)	0.3270***(0.1212)	0.8847***(0.0702)	0.8758***(0.1683)	0.6674***(0.1809)	0.8487***(0.2772)
$\Phi_{2,11}$	NA	−0.2819**(0.1287)	NA	−0.2073*(0.1205)	NA	0.5330(0.3849)
$\Phi_{2,12}$	NA	10.0400(12.6349)	NA	2.2500(16.3773)	NA	18.0614(14.6695)
$\Phi_{2,13}$	NA	1.8566(5.0159)	NA	−0.7693(5.6546)	NA	10.0051(10.7389)
$\Phi_{2,21}$	NA	0.0023*(0.0014)	NA	0.0025(0.0016)	NA	0.0003(0.0063)
$\Phi_{2,22}$	NA	0.3371***(0.0998)	NA	0.0909(0.1742)	NA	−0.6958***(0.1458)
$\Phi_{2,23}$	NA	0.0195(0.0399)	NA	−0.0242(0.0411)	NA	−0.0595(0.1270)
$\Phi_{2,31}$	NA	−0.0029(0.0052)	NA	−0.0036(0.0041)	NA	−0.0492**(0.0224)
$\Phi_{2,32}$	NA	−0.0640(0.4330)	NA	−0.0704(0.4839)	NA	−0.3714(1.0000)
$\Phi_{2,33}$	NA	0.2882**(0.1291)	NA	−0.0134(0.1463)	NA	−1.2614***(0.4601)
$\Psi_{1,11}$	NA	NA	−0.5969***(0.0752)	−0.5495***(0.1424)	0.3871***(0.1018)	0.4251***(0.1046)
Ω_{11}^{-1}	17.4057***(1.2077)	16.6400***(1.5254)	17.1544***(1.1348)	16.7792***(1.3796)	12.8909***(1.3657)	12.2978***(1.3441)
Ω_{21}^{-1}	0.0078(0.0229)	0.0134(0.0314)	0.0121(0.0247)	0.0184(0.0301)	0.0087(0.0167)	0.0034(0.0163)
Ω_{22}^{-1}	0.1743***(0.0102)	0.1582***(0.0096)	0.1568***(0.0113)	0.1506***(0.0114)	0.1354***(0.0130)	0.1157***(0.0109)
Ω_{31}^{-1}	0.1359**(0.0671)	0.1326**(0.0617)	0.1342**(0.0568)	0.1237*(0.0644)	0.0828(0.0528)	0.1033**(0.0501)
Ω_{32}^{-1}	−0.0218(0.0585)	−0.0236(0.0738)	−0.0125(0.0638)	−0.0065(0.0818)	0.0099(0.0597)	0.0084(0.0549)
Ω_{33}^{-1}	0.5390***(0.0433)	0.5176***(0.0470)	0.5242***(0.0431)	0.5222***(0.0450)	0.4347***(0.0394)	0.3825***(0.0378)
ν	NA	NA	NA	NA	5.2255***(1.3534)	3.6079***(0.9566)

Notes: Vector autoregressive (VAR); VAR moving average (VARMA); Quasi-VAR (QVAR); not available (NA). Standard errors are reported in parentheses. *, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively.

Table 3. Model diagnostics; VAR(1), VAR(2), VARMA(1,1), VARMA(2,1), QVAR(1) and QVAR(2) models

	VAR(1)	VAR(2)	VARMA(1,1)	VARMA(2,1)	QVAR(1)	QVAR(2)
C_1	0.6017	0.7779	0.9029	0.8874	0.8660	0.8932
C_2	NA	NA	0.5969	0.5495	NA	NA
C_2 ADF	NA	NA	NA	NA	All stationary	All stationary
C_3	NA	NA	NA	NA	0.8230	0.9010
C_3 ADF	NA	NA	NA	NA	All stationary	All stationary
C_4	NA	NA	NA	NA	0.6791	0.8137
C_4 ADF	NA	NA	NA	NA	All stationary	All stationary
Ljung–Box (20 lags) ϵ_{1t}	26.9600	17.5276	21.0567	19.1193	27.6172	20.4530
Ljung–Box (20 lags) ϵ_{2t}	31.8132**	36.6368**	18.8905	20.6828	15.9674	17.0732
Ljung–Box (20 lags) ϵ_{3t}	27.7176	20.0512	20.7647	19.8994	22.4576	24.7673
Ljung–Box (20 lags) v_{1t}	26.9604	17.5276	21.0567	19.1193	27.6130	20.4530
Ljung–Box (20 lags) v_{2t}	33.5153**	38.6692***	21.0463	23.1736	17.0830	16.9968
Ljung–Box (20 lags) v_{3t}	21.9121	18.7388	19.0656	19.1638	22.5167	20.8977
Ljung–Box (20 lags) u_{1t}	NA	NA	NA	NA	39.6787***	24.4405
Ljung–Box (20 lags) u_{2t}	NA	NA	NA	NA	14.1682	12.6690
Ljung–Box (20 lags) u_{3t}	NA	NA	NA	NA	22.6024	17.4514
Mean LL	−4.7484	−4.5664	−4.6000	−4.5342	**−4.5353**	**−4.4445**
Mean AIC	9.8365	9.6422	9.5586	9.5968	**9.4479**	**9.4362**
Mean BIC	10.2888	10.3206	10.0360	10.3003	**9.9504**	**10.1649**
Mean HQC	10.0198	9.9172	9.7521	9.8819	**9.6516**	**9.7316**
ADF with constant $d_{11,t}$	NA	NA	NA	NA	−9.3502***	−9.1163***
ADF with constant $d_{12,t}$	NA	NA	NA	NA	−8.6100***	−9.8568***
ADF with constant $d_{13,t}$	NA	NA	NA	NA	−10.5493***	−11.1314***
ADF with constant $d_{21,t}$	NA	NA	NA	NA	−8.6100***	−9.8568***
ADF with constant $d_{22,t}$	NA	NA	NA	NA	−8.4884***	−9.3458***
ADF with constant $d_{23,t}$	NA	NA	NA	NA	−9.5828***	−10.4502***
ADF with constant $d_{31,t}$	NA	NA	NA	NA	−10.5493***	−11.1314***
ADF with constant $d_{32,t}$	NA	NA	NA	NA	−9.5828***	−10.4502***
ADF with constant $d_{33,t}$	NA	NA	NA	NA	−10.0015***	−10.3421***

Notes: Vector autoregressive (VAR); VAR moving average (VARMA); Quasi-VAR (QVAR); not available (NA); augmented Dickey–Fuller test statistic (ADF); log-likelihood (LL); Akaike information criterion (AIC); Bayesian information criterion (BIC); Hannan–Quinn criterion (HQC). For QVAR, we summarize the ADF test results for Conditons 2, 3 and 4, by using the terms 'all stationary'. These indicate that for all time series corresponding to Conditons 2, 3 and 4, the unit root null hypothesis of ADF is rejected. Superior statistical performance is indicated by bold numbers. ** and *** indicate significance at the 5% and 1% levels, respectively.

Table 4. Parameter estimates and model diagnostics; restricted QVAR(2) and restricted VARMA(2,1) models

	Restricted QVAR(2)	Restricted VARMA(2,1)		Restricted QVAR(2)	Restricted VARMA(2,1)
c_1	2.5879(1.5854)	0.9350(1.1436)	C_1	0.9090	0.9030
c_2	0.5883***(0.0641)	0.0103(0.0264)	C_2	NA	0.4994
c_3	0.7287***(0.0494)	0.1444**(0.0612)	C_2 ADF	All stationary	NA
$\Phi_{1,11}$	Restricted to zero	0.4450***(0.1274)	C_3	0.9091	NA
$\Phi_{1,12}$	Restricted to zero	Restricted to zero	C_3 ADF	All stationary	NA
$\Phi_{1,13}$	Restricted to zero	Restricted to zero	C_4	0.8276	NA
$\Phi_{1,21}$	0.0072**(0.0034)	Restricted to zero	C_4 ADF	All stationary	NA
$\Phi_{1,22}$	1.6467***(0.0819)	0.9030***(0.0448)	Ljung–Box (20 lags) ϵ_{1t}	27.1409	19.8920
$\Phi_{1,23}$	0.2394***(0.0820)	0.0670***(0.0184)	Ljung–Box (20 lags) ϵ_{2t}	16.7674	21.5212
$\Phi_{1,31}$	Restricted to zero	Restricted to zero	Ljung–Box (20 lags) ϵ_{3t}	48.5889***	20.9798
$\Phi_{1,32}$	Restricted to zero	Restricted to zero	Ljung–Box (20 lags) v_{1t}	27.1409	19.8920
$\Phi_{1,33}$	0.3118***(0.0776)	0.7976***(0.0775)	Ljung–Box (20 lags) v_{2t}	16.7674	21.5212
$\Phi_{2,11}$	Restricted to zero	−0.1883*(0.1140)	Ljung–Box (20 lags) v_{3t}	40.3250***	18.4239
$\Phi_{2,12}$	Restricted to zero	Restricted to zero	Ljung–Box (20 lags) u_{1t}	31.6273**	NA
$\Phi_{2,13}$	Restricted to zero	Restricted to zero	Ljung–Box (20 lags) u_{2t}	15.2545	NA
$\Phi_{2,21}$	Restricted to zero	Restricted to zero	Ljung–Box (20 lags) u_{3t}	25.0686	NA
$\Phi_{2,22}$	−0.6883***(0.0858)	Restricted to zero	Mean LL	−4.4971	−4.6222
$\Phi_{2,23}$	Restricted to zero	Restricted to zero	Mean AIC	9.2961	9.4898
$\Phi_{2,31}$	−0.0298*(0.0153)	Restricted to zero	Mean BIC	9.6981	9.8164
$\Phi_{2,32}$	Restricted to zero	Restricted to zero	Mean HQC	9.4591	9.6222
$\Phi_{2,33}$	−0.8264***(0.0832)	Restricted to zero	ADF with constant $d_{11,t}$	−9.3687	NA
$\Psi_{1,11}$	0.3886***(0.0939)	−0.4994***(0.0852)	ADF with constant $d_{12,t}$	−4.7047	NA
Ω_{11}^{-1}	12.3595***(1.2991)	16.9346***(0.8274)	ADF with constant $d_{13,t}$	−10.6753	NA
Ω_{21}^{-1}	Restricted to zero	Restricted to zero	ADF with constant $d_{21,t}$	−4.7047	NA
Ω_{22}^{-1}	0.1199***(0.0104)	0.1598***(0.0108)	ADF with constant $d_{22,t}$	−9.4062	NA
Ω_{31}^{-1}	0.0918*(0.0489)	0.1313***(0.0475)	ADF with constant $d_{23,t}$	−9.8061	NA
Ω_{32}^{-1}	Restricted to zero	Restricted to zero	ADF with constant $d_{31,t}$	−10.6753	NA
Ω_{33}^{-1}	0.4173***(0.0375)	0.5325***(0.0440)	ADF with constant $d_{32,t}$	−9.8061	NA
ν	3.9974***(0.9251)	NA	ADF with constant $d_{33,t}$	−10.0114	NA

Notes: Quasi-vector autoregressive (QVAR); VAR moving average (VARMA); not available (NA); augmented Dickey–Fuller test statistic (ADF); log-likelihood (LL); Akaike information criterion (AIC); Bayesian information criterion (BIC); Hannan–Quinn criterion (HQC). We summarize the ADF test results for Conditons 2, 3 and 4, by using the terms 'all stationary'. These indicate that for all time series corresponding to Conditons 2, 3 and 4, the unit root null hypothesis of ADF is rejected. Standard errors are reported in parentheses. *, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively.

Table 5. Parameter estimates and model diagnostics; QVAR(2) and restricted QVAR(2) for variable ordering: crude oil, GDP growth and inflation

	QVAR(2)	Restricted QVAR(2)		QVAR(2)	Restricted QVAR(2)
c_1	3.6927(2.9517)	2.5881(1.5854)	C_1	0.8932	0.9090
c_2	0.6997***(0.0635)	0.7287***(0.0494)	C_2	NA	NA
c_3	0.5982***(0.0836)	0.5883***(0.0641)	C_2 ADF	All stationary	All stationary
$\Phi_{1,11}$	0.0434(0.4088)	Restricted to zero	C_3	0.9011	0.9091
$\Phi_{1,12}$	−14.9226(10.5580)	Restricted to zero	C_3 ADF	All stationary	All stationary
$\Phi_{1,13}$	−14.0766(17.4551)	Restricted to zero	C_4	0.8139	0.8276
$\Phi_{1,21}$	−0.0058(0.0156)	Restricted to zero	C_4 ADF	All stationary	All stationary
$\Phi_{1,22}$	0.8487***(0.2772)	0.3118***(0.0776)	Ljung–Box (20 lags) ϵ_{1t}	20.4530	27.1409
$\Phi_{1,23}$	0.6641(0.8138)	Restricted to zero	Ljung–Box (20 lags) ϵ_{2t}	24.6697	48.5888***
$\Phi_{1,31}$	0.0093*(0.0057)	0.0072**(0.0034)	Ljung–Box (20 lags) ϵ_{3t}	17.3403	16.7674
$\Phi_{1,32}$	0.2874***(0.1111)	0.2394***(0.0820)	Ljung–Box (20 lags) v_{1t}	20.4530	27.1409
$\Phi_{1,33}$	1.6014***(0.1649)	1.6467***(0.0819)	Ljung–Box (20 lags) v_{2t}	20.8975	40.3250***
$\Phi_{2,11}$	0.5330(0.3849)	Restricted to zero	Ljung–Box (20 lags) v_{3t}	16.9968	16.7674
$\Phi_{2,12}$	10.0049(10.7389)	Restricted to zero	Ljung–Box (20 lags) u_{1t}	24.4407	31.6274**
$\Phi_{2,13}$	18.0611(14.6693)	Restricted to zero	Ljung–Box (20 lags) u_{2t}	17.4515	25.0687
$\Phi_{2,21}$	−0.0492**(0.0224)	−0.0298*(0.0153)	Ljung–Box (20 lags) u_{3t}	12.6690	15.2545
$\Phi_{2,22}$	−1.2615***(0.4602)	−0.8264***(0.0832)	Mean LL	−4.4445	−4.4971
$\Phi_{2,23}$	−0.3715(1.0000)	Restricted to zero	Mean AIC	9.4362	9.2961
$\Phi_{2,31}$	0.0003(0.0063)	Restricted to zero	Mean BIC	10.1649	9.6981
$\Phi_{2,32}$	−0.0595(0.1270)	Restricted to zero	Mean HQC	9.7316	9.4591
$\Phi_{2,33}$	−0.6958***(0.1458)	0.1199***(0.0104)	ADF with constant $d_{11,t}$	−9.1163	−9.3687
$\Psi_{1,11}$	0.4251***(0.1046)	0.3886***(0.0939)	ADF with constant $d_{12,t}$	−11.1251	−10.6753
Ω_{11}^{-1}	12.2978***(1.3441)	12.3595***(1.2991)	ADF with constant $d_{13,t}$	−9.8592	−4.7047
Ω_{21}^{-1}	0.1033**(0.0501)	0.0918*(0.0489)	ADF with constant $d_{21,t}$	−11.1251	−10.6753
Ω_{22}^{-1}	0.3826***(0.0377)	0.4173***(0.0375)	ADF with constant $d_{22,t}$	−10.3482	−10.0114
Ω_{31}^{-1}	0.0034(0.0163)	Restricted to zero	ADF with constant $d_{23,t}$	−10.3398	−9.8061
Ω_{32}^{-1}	0.0025(0.0166)	Restricted to zero	ADF with constant $d_{31,t}$	−9.8592	−4.7047
Ω_{33}^{-1}	0.1157***(0.0109)	0.1199***(0.0104)	ADF with constant $d_{32,t}$	−10.3398	−9.8061
ν	3.6079***(0.9566)	3.9974***(0.9251)	ADF with constant $d_{33,t}$	−9.3661	−9.4062

Notes: Quasi-vector autoregressive (QVAR); not available (NA); augmented Dickey–Fuller test statistic (ADF); log-likelihood (LL); Akaike information criterion (AIC); Bayesian information criterion (BIC); Hannan–Quinn criterion (HQC). We summarize the ADF test results for Conditons 2, 3 and 4, by using the terms 'all stationary'. These indicate that for all time series corresponding to Conditons 2, 3 and 4, the unit root null hypothesis of ADF is rejected. Standard errors are reported in parentheses. *, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively.

Fig. 1(a) Crude oil y_{1t}

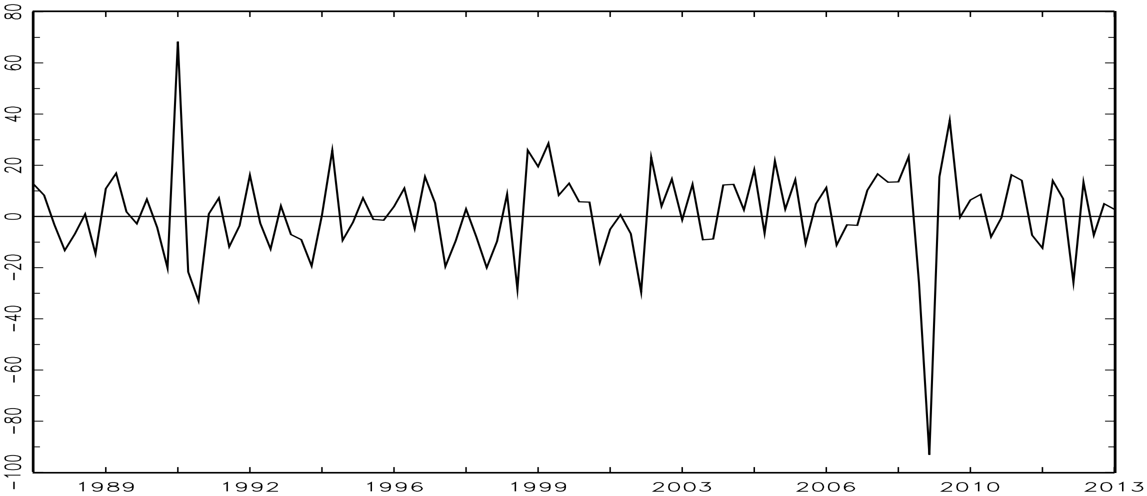

Fig. 1(b) Inflation y_{2t}

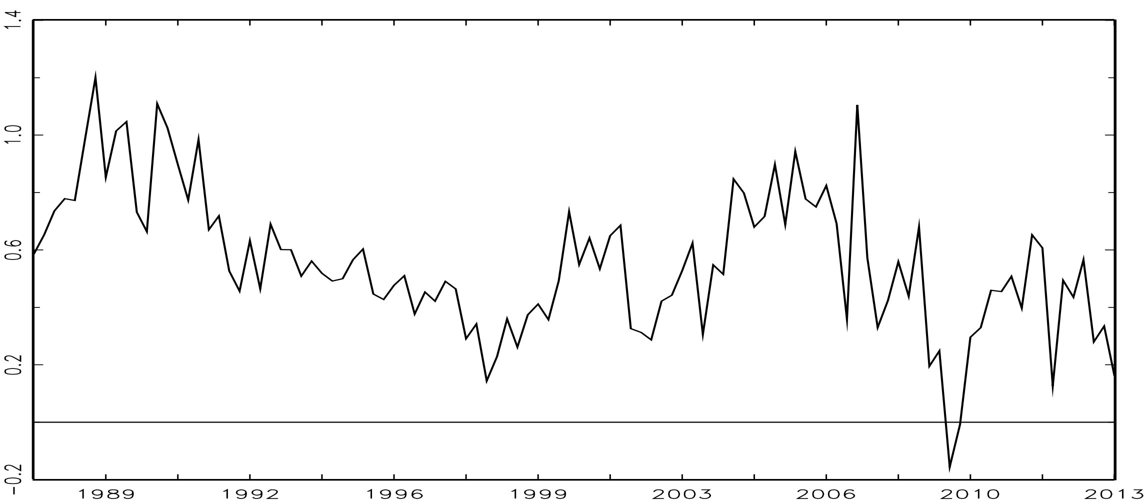

Fig. 1(c) GDP growth y_{3t}

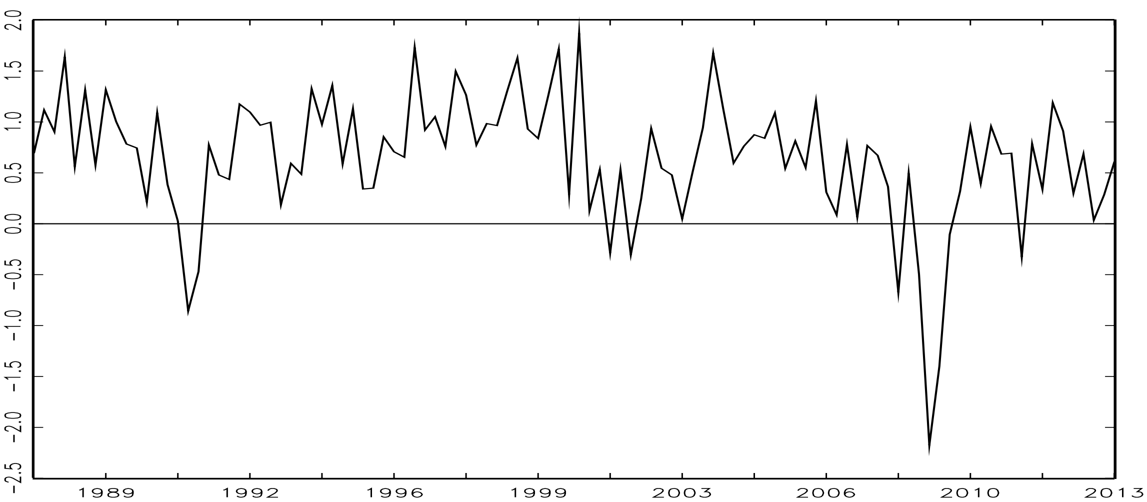

Fig. 1. Dataset for period 1987:Q1 to 2013:Q2.

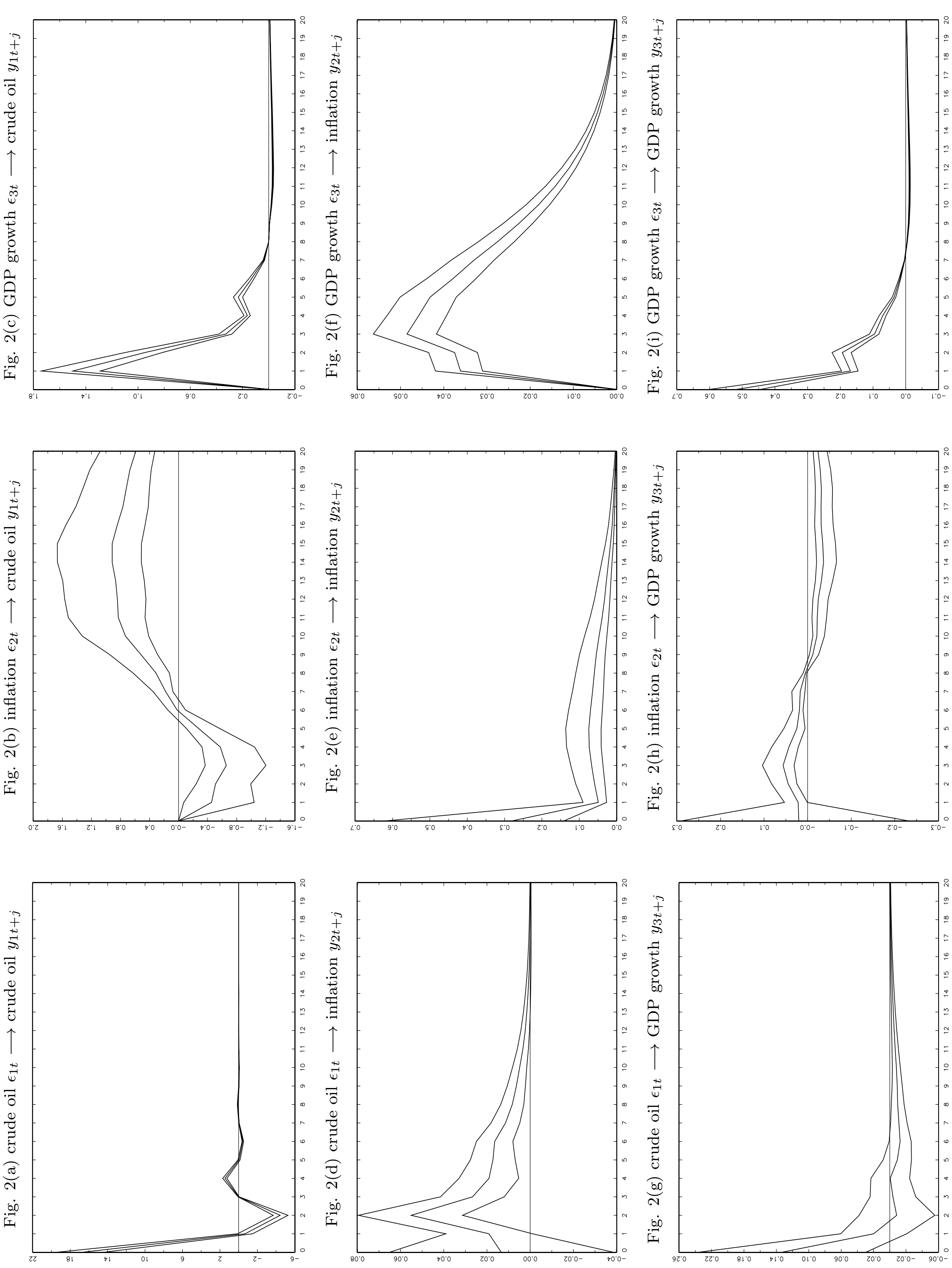

Fig. 2. Impulse response function of QVAR(2) for variable ordering: crude oil, inflation and GDP growth.

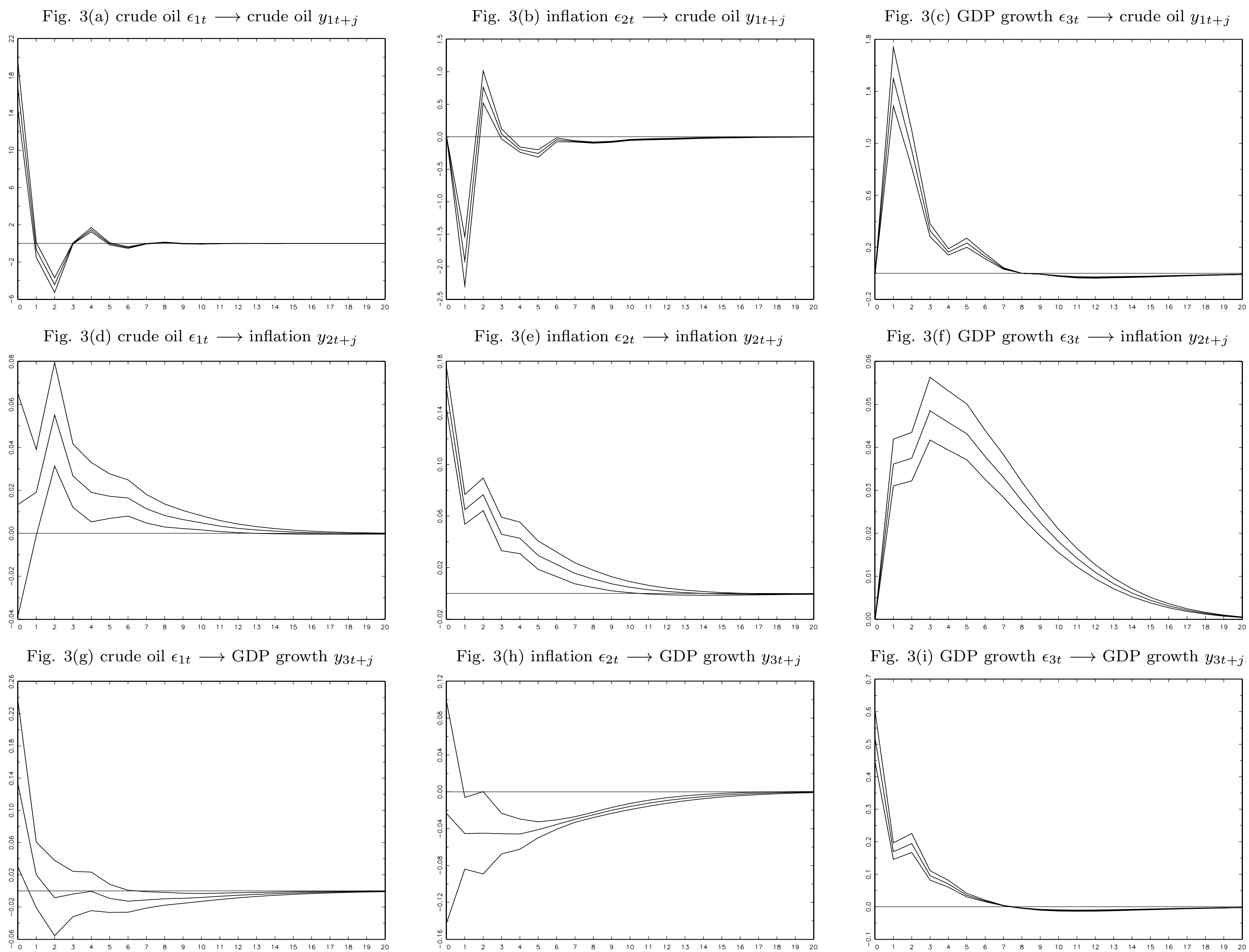

Fig. 3. Impulse response function of VAR(2) for variable ordering: crude oil, inflation and GDP growth.

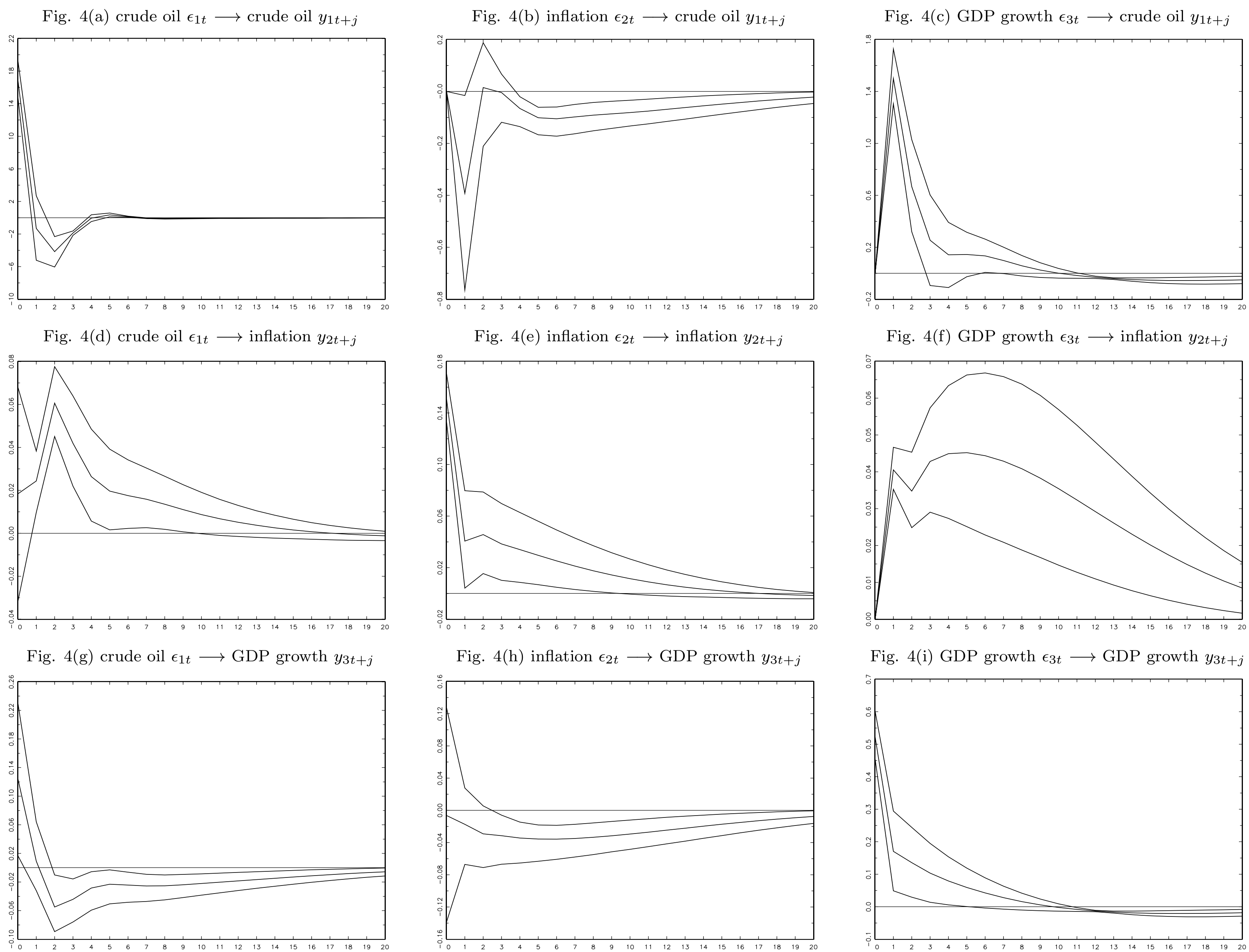

Fig. 4. Impulse response function of VARMA(2,1) for variable ordering: crude oil, inflation and GDP growth.

Fig. 5(a) crude oil $\epsilon_{1t} \longrightarrow$ crude oil y_{1t+j}

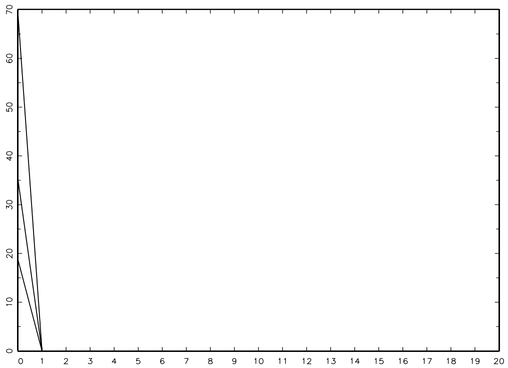

Fig. 5(b) inflation $\epsilon_{2t} \longrightarrow$ crude oil y_{1t+j}

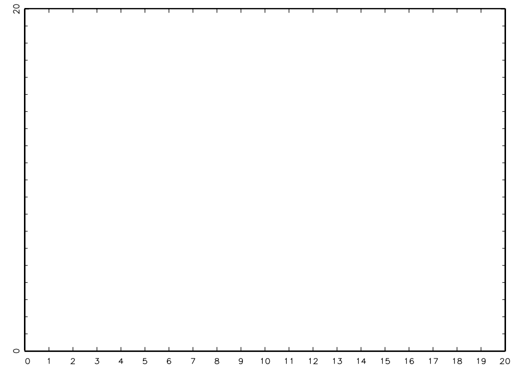

Fig. 5(c) GDP growth $\epsilon_{3t} \longrightarrow$ crude oil y_{1t+j}

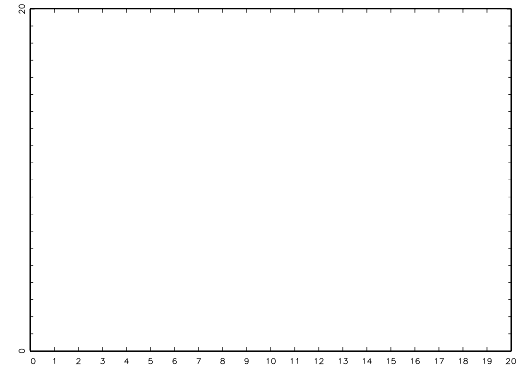

Fig. 5(d) crude oil $\epsilon_{1t} \longrightarrow$ inflation y_{2t+j}

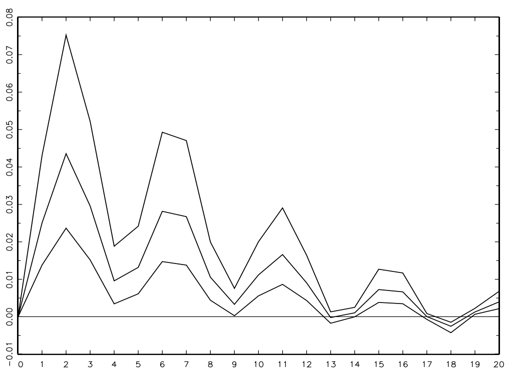

Fig. 5(e) inflation $\epsilon_{2t} \longrightarrow$ inflation y_{2t+j}

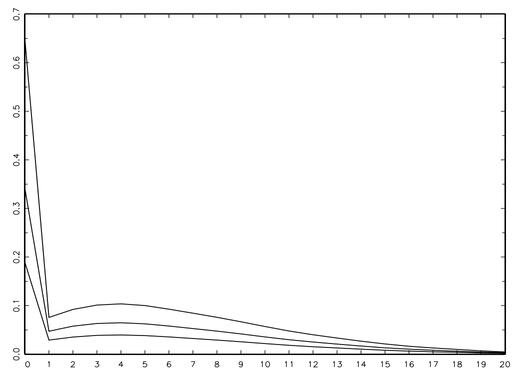

Fig. 5(f) GDP growth $\epsilon_{3t} \longrightarrow$ inflation y_{2t+j}

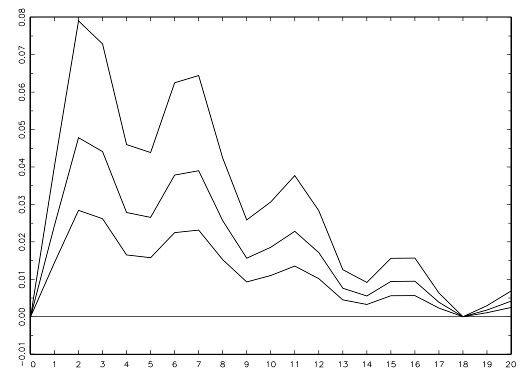

Fig. 5(g) crude oil $\epsilon_{1t} \longrightarrow$ GDP growth y_{3t+j}

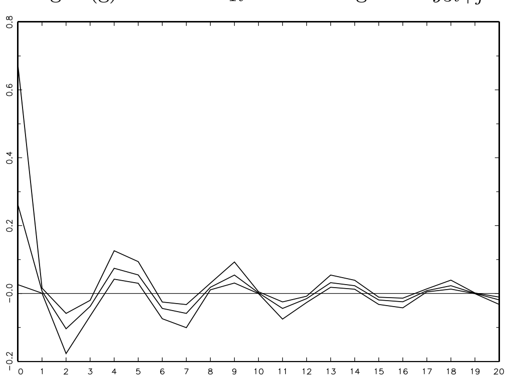

Fig. 5(h) inflation $\epsilon_{2t} \longrightarrow$ GDP growth y_{3t+j}

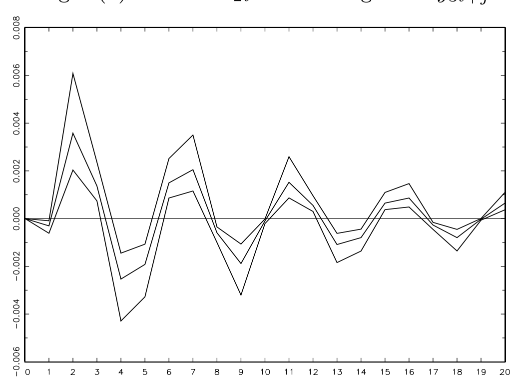

Fig. 5(i) GDP growth $\epsilon_{3t} \longrightarrow$ GDP growth y_{3t+j}

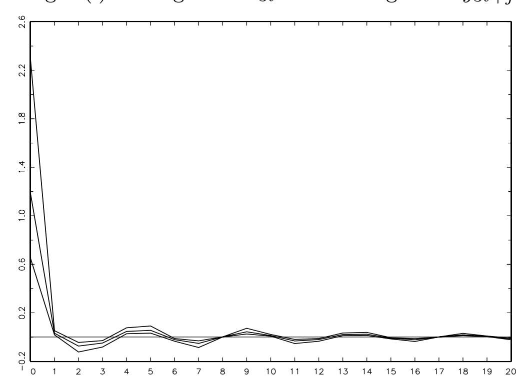

Fig. 5. Impulse response function of restricted QVAR(2) for variable ordering: crude oil, inflation and GDP growth.

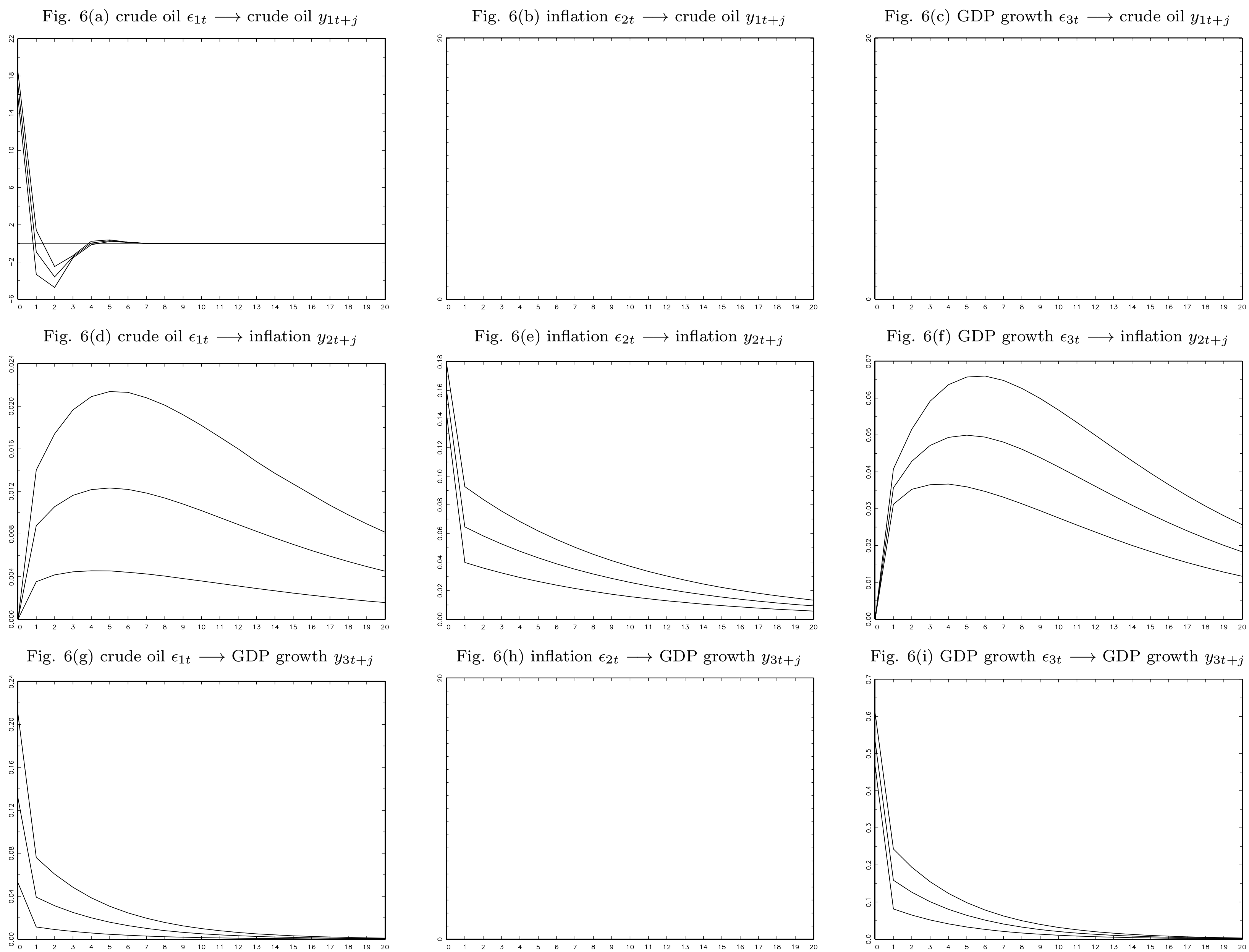

Fig. 6. Impulse response function of restricted VARMA(2,1) for variable ordering: crude oil, inflation and GDP growth.

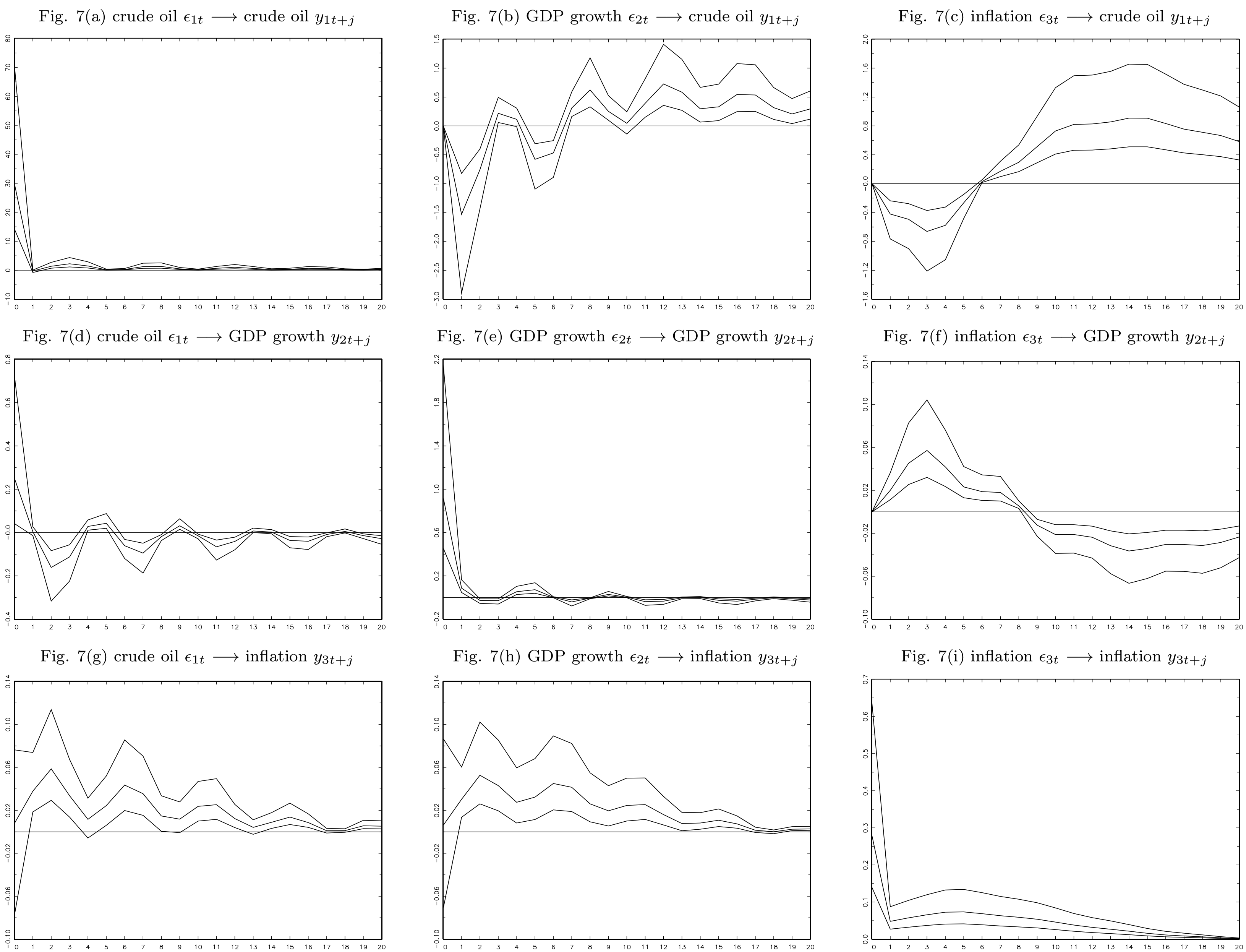

Fig. 7. Impulse response function of QVAR(2) for variable ordering: crude oil, GDP growth and inflation.

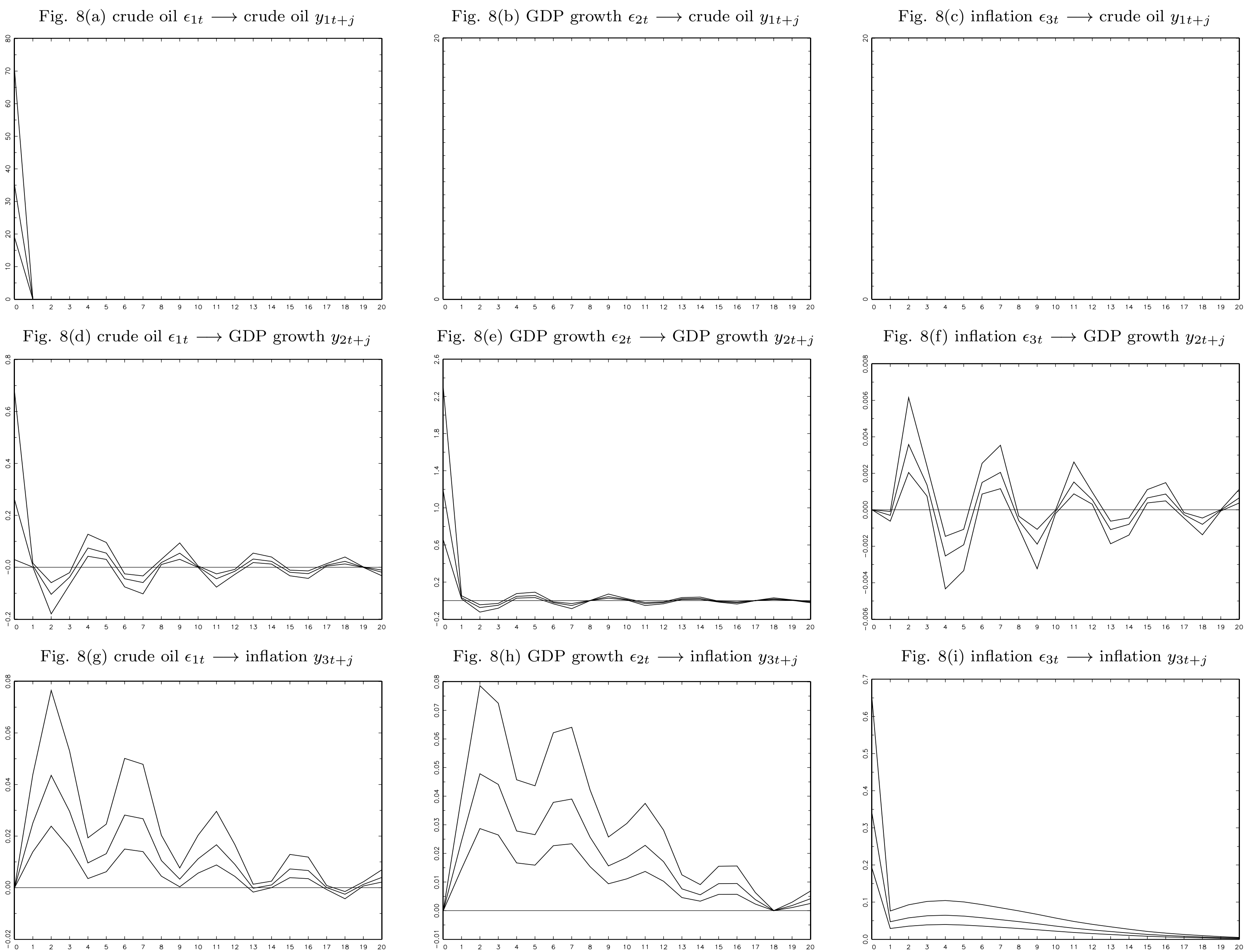

Fig. 8. Impulse response function of restricted QVAR(2) for variable ordering: crude oil, GDP growth and inflation.